A NOTE ON AN INVERSE THEOREM FOR A GENERALISED MODULUS OF SMOOTHNESS

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ABSTRACT. We prove the theorem converse to Jackson's theorem for a modulus of smoothness of the first order generalised by means of an asymmetric operator of generalised translation.

INTRODUCTION

The relation between the modulus of smoothness and best approximation by trigonometric polynomials of a 2π-periodic function is well-known. In the case of non periodic functions there is no such relation between their moduli of smoothness and best approximation by algebraic polynomials. An analogy with the 2π-periodic case takes place if the ordinary modulus of smoothness is replaced by a generalised modulus of smoothness (see e.g. [1, 2, 5]).

In number of papers generalised moduli of smoothness are introduced by means of generalised symmetric operators of translation [3, 4, 5].

In [6], an asymmetric operator of generalised translation is introduced, by means of it a generalised modulus of smoothness of the first order is defined, and the theorem of coincidence of the class of functions defined by that modulus with the class of functions with given order of best approximation by algebraic polynomials is proved.

In the present paper we prove a theorem converse to Jackson's theorem related to that modulus of smoothness.

1. DEFINITIONS

By L_p we denote the set of functions f measurable on the segment $[-1,1]$ such that for $1 \le p < \infty$

$$\|f\|_p = \left(\int_{-1}^{1} |f(x)|^p \, dx \right)^{1/p} < \infty,$$

and for $p = \infty$

$$\|f\|_\infty = \operatorname*{ess\,sup}_{-1 \le x \le 1} |f(x)| < \infty.$$

Denote by $L_{p,\alpha}$ the set of functions f such that $f(x)(1-x^2)^\alpha \in L_p$, and put

$$\|f\|_{p,\alpha} = \left\| f(x)(1-x^2)^\alpha \right\|_p.$$

By $E_n(f)_{p,\alpha}$ we denote best approximation of a function $f \in L_{p,\alpha}$ by algebraic polynomials of degree not greater than $n-1$, in $L_{p,\alpha}$ metrics, i.e.

$$E_n(f)_{p,\alpha} = \inf_{P_n} \|f - P_n\|_{p,\alpha},$$

where P_n are algebraic polynomials of degree not greater than $n-1$.

For a function f we define an operator of generalised translation $\hat{T}_t(f, x)$ by

$$\hat{T}_t(f, x) = \frac{1}{\pi(1 - x^2)} \int_0^\pi \left(1 - \left(x \cos t - \sqrt{1 - x^2} \sin t \cos \varphi\right)^2\right.$$

$$- 2\sin^2 t \sin^2 \varphi + 4\left(1 - x^2\right)\sin^2 t \sin^4 \varphi\bigg)$$

$$\times f(x \cos t - \sqrt{1 - x^2} \sin t \cos \varphi)\, d\varphi.$$

By means of that operator of generalised translation we define the generalised modulus of smoothness by

$$\hat{\omega}(f, \delta)_{p,\alpha} = \sup_{|t| \le \delta} \left\|\hat{T}_t(f, x) - f(x)\right\|_{p,\alpha}.$$

Put $y = \cos t$, $z = \cos \varphi$ in the operator $\hat{T}_t(f, x)$, we denote it by $T_y(f, x)$ and rewrite it in the form

$$T_y(f, x) = \frac{1}{\pi(1 - x^2)} \int_{-1}^1 \left(1 - R^2 - 2\left(1 - y^2\right)\left(1 - z^2\right)\right.$$

$$+ 4\left(1 - x^2\right)\left(1 - y^2\right)\left(1 - z^2\right)^2 \bigg) f(R) \frac{dz}{\sqrt{1 - z^2}},$$

where $R = xy - z\sqrt{1 - x^2}\sqrt{1 - y^2}$.

By $P_\nu^{(\alpha,\beta)}(x)$ $(\nu = 0, 1, \dots)$ we denote the Jacobi's polynomials, i.e. algebraic polynomials of degree ν orthogonal with the weight function $(1 - x)^\alpha(1 + x)^\beta$ on the segment $[-1, 1]$ and normed by the condition $P_\nu^{(\alpha,\beta)}(1) = 1$ $(\nu = 0, 1, \dots)$.

Denote by $a_n(f)$ the Fourier–Jacobi coefficients of a function f, integrable with the weight function $\left(1 - x^2\right)^2$ on the segment $[-1, 1]$, with respect to the system of Jacobi polynomials $\{P_n^{(2,2)}(x)\}_{n=0}^\infty$, i.e.

$$a_n(f) = \int_{-1}^1 f(x) P_n^{(2,2)}(x)\left(1 - x^2\right)^2\, dx \quad (n = 0, 1, \dots).$$

The following properties of the operator T_y are proved in [6].

Lemma 1.1. *Operator T_y has the following properties*
  (1) *The operator $T_y(f, x)$ is linear with respect to f;*
  (2) $T_1(f, x) = f(x)$;
  (3) $T_y\left(P_n^{(2,2)}, x\right) = P_n^{(2,2)}(x) R_n(y)$ $(n = 0, 1, \dots)$,
     where $R_n(y) = P_{n+2}^{(0,0)}(y) + \frac{3}{2}\left(1 - y^2\right) P_n^{(2,2)}(y)$;
  (4) $T_y(1, x) = 1$;
  (5) $a_k\left(T_y(f, x)\right) = R_k(y) a_k(f)$ $(k = 0, 1, \dots)$.

Lemma 1.2. *Let given numbers p, α, r and λ be such that $1 \le p \le \infty$, $0 < \lambda < 2$;*

$$\frac{1}{2} < \alpha \le 1 \qquad \text{for } p = 1,$$

$$1 - \frac{1}{2p} < \alpha < \frac{3}{2} - \frac{1}{2p} \quad \text{for } 1 < p < \infty,$$

$$1 \le \alpha < \frac{3}{2} \qquad \text{for } p = \infty.$$

Let $f \in L_{p,\alpha}$. Then

$$E_n(f)_{p,\alpha} \le \frac{C_1}{n^\lambda}$$

if and only if

$$\hat{\omega}(f, \delta)_{p,\alpha} \le C_2 \delta^\lambda,$$

where constants C_1 and C_2 do not depend on f, n and δ.

The lemma is proved in [6].

2. THE CONVERSE THEOREM

Now we formulate our result.

Theorem 2.1. *Let given numbers p, α and λ be such that $1 \le p \le \infty$, $0 < \lambda < 2$;*

$$\frac{1}{2} < \alpha \le 1 \qquad for \ p = 1,$$

$$1 - \frac{1}{2p} < \alpha < \frac{3}{2} - \frac{1}{2p} \quad for \ 1 < p < \infty,$$

$$1 \le \alpha < \frac{3}{2} \qquad for \ p = \infty.$$

If $f \in L_{p,\alpha}$, then the following inequality holds

$$\hat{\omega}\left(f, \frac{1}{n}\right)_{p,\alpha} \le \frac{C_1}{n^2} \sum_{\nu=1}^{n} \nu E_\nu(f)_{p,\alpha},$$

where the constant C does not depend on f and n.

Proof. Let $P_n(x)$ be the polynomial of degree not greater than $n-1$ such that

$$\|f - P_n\|_{p,\alpha} = E_n(f)_{p,\alpha} \quad (n = 1, 2, \dots),$$

and

$$Q_k(x) = P_{2^k}(x) - P_{2^{k-1}}(x) \quad (k = 1, 2, \dots),$$

$Q_0(x) = P_1(x)$.

For given n we chose the positive integer N such that

$$\frac{n}{2} < 2^N \le n + 1.$$

By the proof of Lemma 1.2 given in [6] it follows that

$$\hat{\omega}\left(f, \frac{1}{n}\right)_{p,\alpha} \le C_2\left(E_{2^N}(f)_{p,\alpha} + \frac{1}{n^2}\sum_{\mu=1}^{N} 2^{2\mu}\|Q_k\|_{p,\alpha}\right)$$

$$\le 2C_2\left(E_{2^N}(f)_{p,\alpha} + \frac{1}{n^2}\sum_{\mu=1}^{N} 2^{2\mu}\left(E_{2^\mu}(f)_{p,\alpha} + E_{2^{\mu-1}}(f)_{p,\alpha}\right)\right)$$

$$\le 4C_2\left(E_{2^N}(f)_{p,\alpha} + \frac{1}{n^2}\sum_{\mu=0}^{N-1} 2^{2(\mu+1)}E_{2^\mu}(f)_{p,\alpha}\right)$$

$$\le \frac{C_3}{n^2}\sum_{\mu=0}^{N} 2^{2(\mu+1)}E_{2^\mu}(f)_{p,\alpha}.$$

Considering that for $\mu \ge 1$ we have

$$\sum_{\nu=2^{\mu-1}}^{2^\mu-1} \nu E_\nu(f)_{p,\alpha} \ge E_{2^\mu}(f)_{p,\alpha}\, 2^{2(\mu-1)},$$

it follows that

$$\hat{\omega}\left(f, \frac{1}{n}\right)_{p,\alpha} \le \frac{C_4}{n^2}\left(2^2 E_1\left(f\right)_{p,\alpha} + \sum_{\mu=1}^{N}\sum_{\nu=2^{\mu-1}}^{2^\mu-1}\nu E_\nu\left(f\right)_{p,\alpha}\right)$$
$$\le \frac{C_5}{n^2}\sum_{\nu=1}^{n}\nu E_\nu\left(f\right)_{p,\alpha}.$$

Theorem 2.1 is proved. $\qquad\qquad\square$

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